On the discrete analogs of the Tzitzeica and the Sawada–Kotera equations

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We study an integrable P∆E which defines a discretization of the Tzitzeica equation. The integrability follows from the Lax representation which is interpreted geometrically as the Gauss equations for the deformed discrete affine spheres. The higher continuous symmetry is an inhomogeneous lattice equation of Bogoyavlensky type which serves as a discrete analog of the SK equation. The r-matrix formulation and bilinear equations are presented.

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Tzitzeica equation

$$H_{,xy} = e^{H} - e^{-2H}$$

- applications in differential geometry
- spectral problem is of 3-rd order
- Bäcklund transformation is of 2-nd order
- the higher symmetry is of 5-th order; it is related via Miura map to the **Sawada–Kotera** equation

$$U_{,\tau} = U_{,xxxxx} + 5UU_{,xxx} + 5U_{,x}U_{,xx} + 5U^{2}U_{,x}$$

[1] G. Tzitzeica. Sur une nouvelle classe de surfaces. *Rendiconti del Circolo Matematico di Palermo* ***25:1*** (1907) 180–187.

[2] K. Sawada, T. Kotera. *Progr. Theor. Phys.* ***51:5*** (1974) 1355–1367.

[3] R.K. Dodd, R.K. Bullough. *Proc. Roy. Soc. London A* ***352*** (1977) 481–503.

[4] A.V. Zhiber, A.B. Shabat. *Soviet Phys. Doklady* ***24*** (1979) 607.

[5] A.V. Mikhailov. *Soviet Phys. JETP Lett.* ***30*** (1979) 414–418.

SUR UNE NOUVELLE CLASSE DE SURFACES.

Par M. **Georges Tzitzéica** (Bucarest).

Adunanza del 10 novembre 1907.

9. Supposons maintenant $U \neq 0$, $V \neq 0$. En remplaçant h par $h_1 = h\sqrt[3]{UV}$, et en faisant le changement de variables $\alpha = \int \sqrt[3]{U}\,du$, $\beta = \int \sqrt[3]{V}\,dv$ on obtient le système

$$(13) \qquad \begin{cases} \dfrac{\partial^2 \omega}{\partial \alpha^2} = \dfrac{1}{h}\dfrac{\partial h}{\partial \alpha}\dfrac{\partial \omega}{\partial \alpha} + \dfrac{1}{h}\dfrac{\partial \omega}{\partial \beta}, \\[2ex] \dfrac{\partial^2 \omega}{\partial \beta^2} = \dfrac{1}{h}\dfrac{\partial \omega}{\partial \alpha} + \dfrac{1}{h}\dfrac{\partial h}{\partial \beta}\dfrac{\partial \omega}{\partial \beta}, \\[2ex] \dfrac{\partial^2 \omega}{\partial \alpha \partial \beta} = h\omega, \end{cases}$$

où h est une intégrale de l'équation

$$(14) \qquad \frac{\partial^2 \log h}{\partial \alpha \partial \beta} = h - \frac{1}{h^2}.$$

Outline of the talk

$$H_{,xy} = e^{H} - e^{-2H} \qquad \text{(Tz)}$$

$\uparrow$ continuous limit

$$hh_{12}(c^{-1}h_1h_2 - h_1 - h_2) + h_{12} + h - c = 0 \qquad \text{(dTz)}$$

$\rightsquigarrow$ higher symmetry

$$h_{,t} = \ldots$$

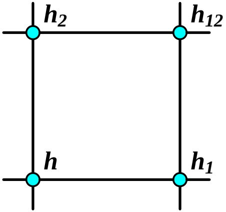
$\Downarrow$ Miura map

$$u_{,t} = u^2(u_{11}u_1 - u_{\overline{1}}u_{\overline{11}}) - u(u_1 - u_{\overline{1}}) \qquad \text{(dSK)}$$

$\downarrow$ continuous limit

$$U_{,\tau} = U_{,xxxxx} + 5UU_{,xxx} + 5U_{,x}U_{,xx} + 5U^2U_{,x} \qquad \text{(SK)}$$

Notations



➢ (dTz) is a partial difference equation with two independent variables. It relates the values of h in the vertices of any elementary plaquette:

$$Q(h, h_1, h_2, h_{12}) = 0,$$
$$h = h(n_1, n_2), \quad h_1 = h(n_1 + 1, n_2), \quad \dots$$

➢ (dSK) is a partial differential-difference equation with one discrete and one continuous variable:

$$u_{,t} = f(u_{11}, u_1, u, u_{\overline{1}}, u_{\overline{11}}), \qquad u = u(n_1, t), \qquad u_{\overline{1}} = u(n_1 - 1, t), \quad \dots$$

➢ later on, we use also the alternative notanion

$$u_{,t} = f(u_2, u_1, u, u_{-1}, u_{-2}), \qquad u = u(n, t), \qquad u_k = u(n + k, t).$$

Bobenko–Schief discretizations

The following discretizations were introduced in [6, 7, 8]:

$$hh_{12}(h_1h_2 - h_1 - h_2) + h_{12} + h - 1 - \frac{AB}{h}h_1h_2h_{12} = 0, \qquad A_2h = Ah_1, \qquad B_1h = Bh_2 \tag{BS-1}$$

and

$$(\gamma - 1)(h_{12} - 1)(h - 1)h + h_{12}(h_1A - h)(h_2B - h) - \gamma h_{12}(h_1 - 1)(h_2 - 1)h^2 = 0, \qquad A_2h = Ah_1, \qquad B_1h = Bh_2. \tag{BS-2}$$

[6] W.K. Schief. In: P. Clarkson, F. Nijhoff (eds), Symmetries and Integrability of Difference Equations. LMS, Lecture Note Series 255, Cambridge UP (1999) 137–148.

[7] A.I. Bobenko, W.K. Schief. In: A. Bobenko, R. Seiler (eds), Discrete Integrable Geometry and Physics. Oxford UP (1999) 113–138.

[8] A.I. Bobenko, W.K. Schief. *Exp. Math.* **8** (1999) 261–280.

- (BS-2) defines a Bäcklund transformation for (BS-1).
- Both versions admit the reduction $A = B = 0$:

$$\begin{array}{ccc} (\text{BS}-1) & \xrightarrow{A=B=0} & (\text{dL}) \\ & & \uparrow c=0 \\ (\text{BS}-2) & \xrightarrow{A=B=0} & (\text{dTz}) \end{array}$$

- Recall that the **discrete Liouville** equation

$$hh_{12}(h_1 - 1)(h_2 - 1) = (h - 1)(h_{12} - 1) \tag{dL}$$

is linearizable: its explicit general solution is obtained via the substitution $h = \frac{\tau_1 \tau_2}{\tau \tau_{12}}$ from solutions of the discrete wave equation $\tau_{12} - \tau_1 - \tau_2 + \tau = 0$:

$$h = \frac{(a_1 - b)(a - b_2)}{(a - b)(a_1 - b_2)}, \quad a = a(n_1), \quad b = b(n_2).$$

Continuous limit

- (dTz) → (Tz), $\varepsilon \to 0$:

$$c = 1 + \alpha\varepsilon^6, \quad h(n_1, n_2) = 1 + \beta\varepsilon^2 v(x, y), \quad x = \varepsilon n_1, \quad y = \varepsilon n_2.$$

Terms up to ε^5 vanish identically. The equation appears at ε^6:

$$\beta^2(v v_{xy} - v_x v_y) = 2\beta^3 v^3 - 2\alpha.$$

This is (Tz) up to the change $v = e^H$ and scalings.

- If $c \equiv 1$ then $\alpha = 0$, the Liouville equation case.

- (dSK) → (SK), $\varepsilon \to 0$:

$$u(n, t) = \frac{1}{3} + \frac{\varepsilon^2}{9} U\Big(x - \frac{4}{9}\varepsilon t, \tau + \frac{2\varepsilon^5}{135} t\Big), \quad x = \varepsilon n.$$

Spectral problem in the continuous case

Gauss equations for the affine spheres $\psi : \mathbb{R}^2 \to \mathbb{R}^3$ with the indefinite Blaschke metrics:

$$\begin{aligned} \psi_{xx} &= H_x \psi_x + \lambda e^{-H} \psi_y, \\ \psi_{xy} &= e^H \psi, \\ \psi_{yy} &= \lambda^{-1} e^{-H} \psi_x + H_y \psi_y. \end{aligned} \tag{1}$$

The geometry:

1) x, y — **asymptotic coordinates** (ψ_{xx}, ψ_{yy} lie in the tangent plane);

2) the Tzitzeica condition: **affine normals** ψ_{xy} meet in the origin.

(Tz) appears as the compatibility conditions (Mainardi–Codazzi equations) which can be written as the zero curvature representation:

$$\Psi_x = U\Psi, \quad \Psi_y = V\Psi \quad \Rightarrow \quad U_y - V_x = [V, U],$$

$$\Psi = \begin{pmatrix} \psi \\ \psi_x \\ \psi_y \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & H_x & \lambda e^{-H} \\ e^H & 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 0 & 1 \\ e^H & 0 & 0 \\ 0 & \lambda^{-1} e^{-H} & H_y \end{pmatrix}.$$

Discrete indefinite affine spheres

Analogously, system (BS-1) is the consistency condition for the discrete Gauss equations ($\psi : \mathbb{Z}^2 \to \mathbb{R}^3$):

$$\begin{aligned}
\psi_{11} - \psi_1 &= \frac{h_1 - 1}{h_1(h-1)}(\psi_1 - \psi) + \frac{\lambda A}{h-1}(\psi_{12} - \psi_1), \\
\psi_{12} + \psi \ &= h(\psi_1 + \psi_2), \\
\psi_{22} - \psi_2 &= \frac{h_2 - 1}{h_2(h-1)}(\psi_2 - \psi) + \frac{\lambda^{-1} B}{h-1}(\psi_{12} - \psi_2).
\end{aligned}$$

This can be written as the discrete zero curvature representation with respect to $\Psi = (\psi, \psi_1, \psi_2)^T$:

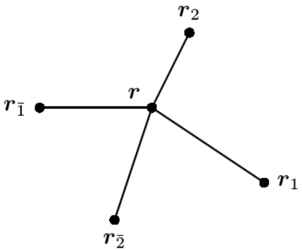
$$\Psi_1 = L\Psi, \quad \Psi_2 = M\Psi \quad \Rightarrow \quad L_2 M = M_1 L.$$

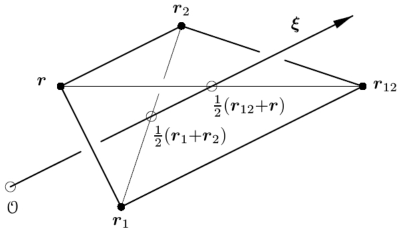
The geometry behind this system is the following.

1) discrete asymptotic net. Any five neighbour points

$$\psi, \quad \psi_1, \quad \psi_{\overline{1}}, \quad \psi_2, \quad \psi_{\overline{2}}$$

are coplanar (1st and 3rd equations). The figures are taken from [8].





2) discrete affine Lorentz harmonic net. The discrete affine normals attached to the centers of plaquettes meet in the origin (2nd equation):

$$\psi_{12}-\psi_1-\psi_2+\psi = k(\psi_{12}+\psi_1+\psi_2+\psi).$$

In the (BS-2) case the geometry is not so illustrative and we will consider only reduction (dTz).

Spectral problem for (dTz)

Theorem 1. (dTz) is the compatibility condition for the Gauss equations which define the following deformation of the discrete affine spheres:

$$\begin{aligned}
\psi_{11}-\mu\psi_1 &= \frac{h_1-c}{h_1(h-c)}(\psi_1-\mu\psi)+\frac{c-\mu}{h-c}(\psi_{12}-\nu\psi_1),\\
\psi_{12}+\psi &= h(\psi_1+\psi_2),\\
\psi_{22}-\nu\psi_2 &= \frac{h_2-c}{h_2(h-c)}(\psi_2-\nu\psi)+\frac{c-\nu}{h-c}(\psi_{12}-\mu\psi_2)
\end{aligned}$$

where $\quad \mu=c-(c+1)\lambda, \quad \nu=c-(c-1)\lambda^{-1}$.

Property **2)** is preserved and **1)** is fulfilled after the change

$$\widetilde{\psi}(n_1,n_2)=\mu^{-n_1}\nu^{-n_2}\psi(n_1,n_2).$$

Certainly, it is possible to rewrite everything in terms of $\widetilde{\psi}$, but then property 2) will be distorted.

A continuous higher symmetry

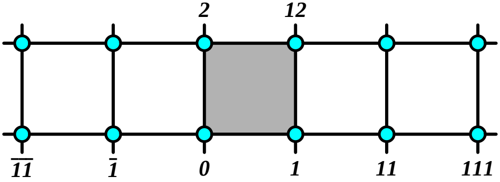
Theorem 2. Equation (dTz) $Q=0$ is consistent with the following differential-difference equation (along any of two directions in $\mathbb{Z}^2$)

$$h_t = \frac{h(c-h)}{h_1 h h_{\bar{1}} - c}\left(\frac{h(c-h_1)(c-h_{\bar{1}})(h_{11}h_1 - h_{\bar{1}}h_{\overline{11}})}{(h_{11}h_1 h - c)(h h_{\bar{1}} h_{\overline{11}} - c)} - h_1 + h_{\bar{1}}\right). \qquad (2)$$

Consistency means that

$$D_t(Q)|_{Q=0} = 0.$$

A direct check of this identity involves 5 plaquettes:



Difference analog of the Sawada–Kotera equation

How (2) is derived?

Elimination of ψ_2 and ψ_{12} yields the difference 3rd order spectral problem

$$u\psi_{111} + \psi_{11} = \lambda(\psi_1 + u\psi), \quad u := \frac{h_{11}(c - h_1)}{h_{11}h_1h - c}.$$

Certainly, a similar equation holds for the second coordinate direction. However, from now we forget about it and **change the notations:**

$$u\psi_3 + \psi_2 = \lambda(\psi_1 + u\psi). \tag{3}$$

The change $h \mapsto u$ is a Miura type transformation: $h = \phi/\phi_1$ where $\psi = \phi$ is a particular solution at $\lambda = 1/c$. Actually, two transformations exist:

$$M^- : \quad u = \frac{h_2(c - h_1)}{h_2h_1h - c}, \qquad M^+ : \quad \hat{u} = \frac{(c - h_1)h}{h_2h_1h - c}$$

which reflects the invariance with respect to the change $h \to h^{-1}$, $c \to c^{-1}$.

Equation (3) is a spectral problem with the operator L equal to the quotient of two difference operators (cf [9] in the continuous case):

$$L\psi = \lambda\psi, \quad L = (T+u)^{-1}(uT+1)T^2$$

where T is the shift operator

$$T^k: \ \psi(n) \mapsto \psi(n+k) = \psi_k.$$

The isospectral deformations are defined as usual: $\psi_{,t} = A\psi$.

Theorem 3. Transformations $M^{\pm}$ map equation (2) into equation

$$u_{,t} = u^2(u_2u_1 - u_{-1}u_{-2}) - u(u_1 - u_{-1}) \tag{dSK}$$

which possesses the Lax representation $L_{,t} = [A, L]$ with

$$A = (u_{-1}T + 1 - u_{-1}u_{-2} + u_{-2}T^{-1})(T - T^{-1}).$$

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[9] I. Krichever. *Phys. D* ***87:1–4*** (1995) 14–19.

Equation (dSK) appeared in [10] for the first time. Recall that both flows

$$u_{,t'} = u(u_1 - u_{-1}) \qquad \text{and} \qquad u_{,t''} = u^2(u_2 u_1 - u_{-1} u_{-2})$$

are very well known: the first one is the Volterra lattice [11, 12] and the second one is the modified Bogoyavlensky lattice [13, 14, 15]. The continuous limit is KdV equation $U_t = U_{xxx} + 6UU_x$ in both cases.

However, these flows belong to the different hierarchies, that is $\partial_{t'}$ and $\partial_{t''}$ do not commute. Hence one should not expect that their linear combination (dSK) remains integrable. But it does.

[10] S. Tsujimoto, R. Hirota. *J. Phys. Soc. Jpn.* ***65*** (1996) 2797–2806.

[11] V.E. Zakharov, S.L. Musher, A.M. Rubenchik. *JETP Letters* ***19:5*** (1974) 249–253.

[12] S.V. Manakov. *JETP Letters* ***67:2*** (1974) 543–555.

[13] K. Narita. *J. Phys. Soc. Jpn* ***51:5*** (1982) 1682–1685.

[14] Y. Itoh. *Prog. Theor. Phys.* ***78*** (1987) 507–510.

[15] O.I. Bogoyavlensky. *Russian Math. Surveys* ***46:3*** (1991) 1–64.

r-matrix description

The hierarchy of higher commuting flows is given by the formula ($t = t_1$)

$$L_{,t_s} = [\pi_+(L^s), L]$$

where the projection π_+ corresponds to the decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ of Lie algebra into sum of two Lie subalgebras:

$$\mathfrak{g} = \Big\{ \sum_{j<\infty} g^{(j)} T^j \Big\}$$

$$\mathfrak{g}_+ = \big\{ F(T - T^{-1}) |\; F = F^\dagger \big\}, \qquad \mathfrak{g}_- = \Big\{ \sum_{j\le 0} h^{(j)} T^j \Big\}.$$

The conserved quantities are $\sum_n H_n^{(s)}, \quad H^{(s)} = \mathrm{coef}_0(L^s)$:

$$H^{(0)} = \log u, \quad H^{(1)} = u - u_{-1} u u_1, \quad \dots .$$

However, a Hamiltonian structure is not recovered till now. ☹

Some generalizations

➢ A rich family of discretizations of (SK) is related with the spectral problems

$$u\psi_{m+l} + \psi_l = \lambda(\psi_m + u\psi)$$

where $m, l > 0$ are coprimes.

At $l = 1$ the following equations appear:

$$u_{,t} = u^2(u_m \cdots u_1 - u_{-1} \cdots u_{-m}) - u(u_{m-1} \cdots u_1 - u_{-1} \cdots u_{1-m}).$$

This is again a sum of two modified Bogoyavlensky lattices.

At $l > 1$ the simplest equation in the hierarchy is not evolutionary:

$$u_m f_m = u f_{m+l}, \quad u_{,t} = u^2(f - f_{m+l}) + f_m - f_l, \qquad (\mathsf{dSK}_{l,m})$$

but some higher flows still can be cast into the evolutionary form.

➢ An intersecting example is related to the spectral problem

$$u_{-3}\psi_{-3}+\psi_{-1}=\lambda(\psi_1+u\psi_3).$$

The associated equation

$$u_{,t}=u(w_3-w_2+w_1-w_{-1}+w_{-2}-w_{-3}-u_2+u_{-2}),\quad w:=u_1uu_{-1}\ \text{(dKK)}$$

is a discrete analog of Kaup–Kupershmidt equation

$$U_{,\tau}=U_{,xxxxx}+5UU_{,xxx}+\frac{25}{2}U_{,x}U_{,xx}+5U^2U_{,x}.$$

➢ Multifield examples can be obtained from the linear problems $P\psi=\lambda Q\psi$ with generic P,Q. For instance, the choice

$$K^\dagger\psi=\lambda K\psi,\quad K=uT^3+v_{-1}T+T^{-1}$$

brings to the lattice

$$u_{,t}=-u(u_2v_2-u_1v_1+u_{-1}v_{-1}-u_{-2}v_{-2}-v_1+v_{-1}),$$
$$v_{,t}=-v(u_1v_1-u_{-1}v_{-1})+u_2u_1-u_{-1}u_{-2}+u_1-u_{-1}.$$

Its 4-th order symmetry admits the reduction $v=0$ to (dKK).

Bilinear equations

Continuous case. The substitution $h = -2(\log\tau)_{xy}$ maps (Tz)

$$hh_{xy} - h_x h_y = h^3 - 1$$

into the trilinear equation

$$4\det\begin{pmatrix}\tau_{yy} & \tau_{xyy} & \tau_{xxyy}\\ \tau_y & \tau_{xy} & \tau_{xxy}\\ \tau & \tau_x & \tau_{xx}\end{pmatrix} = \tau^3. \tag{4}$$

There are two more simple bilinear equations which are consistent with (4):

$$\begin{aligned}3(\tau_{xy}\tau_{xx} - \tau_x\tau_{xxy}) &= \tau_y\tau_{xxx} - \tau\tau_{xxxy},\\ 3(\tau_{xy}\tau_{yy} - \tau_y\tau_{xyy}) &= \tau_x\tau_{yyy} - \tau\tau_{xyyy}.\end{aligned} \tag{5}$$

These appears from the conservation laws

$$\left(\frac{h_{xx}}{h}\right)_y = 3D_x(h), \quad \left(\frac{h_{yy}}{h}\right)_x = 3D_y(h).$$

Indeed, a single integration is possible after the substitution:

$$\frac{h_{xx}}{h} = -6(\log\tau)_{xx} + a(x), \quad \frac{h_{yy}}{h} = -6(\log\tau)_{yy} + b(y).$$

We may set $a = b = 0$, since the τ-function in (4) is defined up to the multiplication by $A(x)$ and $B(y)$. Replacing h by τ yields (5).

Discretization (BS-1). The substitution

$$h = \frac{\tau_1\tau_2}{\tau\tau_{12}}, \quad A = a\frac{\tau_1^2}{\tau\tau_{11}}, \quad B = b\frac{\tau_2^2}{\tau\tau_{22}}$$

yields the trilinear equation

$$\det\begin{pmatrix}\tau_{22} & \tau_{122} & \tau_{1122}\\ \tau_2 & \tau_{12} & \tau_{112}\\ \tau & \tau_1 & \tau_{11}\end{pmatrix} = ab\tau_{12}^3.$$

The bilinear ones are not known.

Discretization (dTz). The substitution $h = \dfrac{\tau_1\tau_2}{\tau\tau_{12}}$ yields

$$c^{-1}\tau_{22}\tau_{12}\tau_{11} + \tau\tau_{122}\tau_{112} + \tau_1\tau_2\tau_{1122} = \tau_{11}\tau_2\tau_{122} + \tau_1\tau_{22}\tau_{112} + c\tau\tau_{12}\tau_{1122}$$

or

$$\det\begin{pmatrix}\tau_{22} & \tau_{122} & \tau_{1122}\\ \tau_2 & c^{-1}\tau_{12} & \tau_{112}\\ \tau & \tau_1 & \tau_{11}\end{pmatrix} = (c - c^{-1})\tau\tau_{12}\tau_{1122}.$$

Both versions give (4) under the continuous limit.

Complementary bilinear equations

$$\tau_{11}\tau_{12} - c\tau_1\tau_{112} = c\tau\tau_{1112} - \tau_{111}\tau_2,$$
$$\tau_{12}\tau_{22} - c\tau_2\tau_{122} = c\tau\tau_{1222} - \tau_{222}\tau_1$$

are derived from the conservation laws

$$\frac{u_2}{u} = \frac{h}{h_{11}}, \quad u := \frac{h_{11}(c - h_1)}{h_{11}h_1h - c}; \qquad \frac{v_1}{v} = \frac{h}{h_{22}}, \quad v := \frac{h_{22}(c - h_2)}{h_{22}h_2h - c}.$$

A single integration is possible after the substitution:

$$u = a(n_1)\frac{\tau\tau_{111}}{\tau_{11}\tau_1}, \quad v = b(n_2)\frac{\tau\tau_{222}}{\tau_{22}\tau_2},$$

Again, we set $a = b = 1$, since the τ-function is defined up to the multiplication by $A(n_1)$ and $B(n_2)$. Replacing h by τ yields the bilinear equations.

3-soliton solution of (dTz):

$$\tau = q^{-n_1 n_2}(1+e_1+e_2+e_3+A_{12}e_1e_2+A_{13}e_1e_3+A_{23}e_2e_3+A_{12}A_{13}A_{23}e_1e_2e_3)$$

where

$e_i = \alpha_i^{n_1}\beta_i^{n_2}\gamma_i$	an analog of $\exp(\alpha x + \beta y + \gamma)$
$\dfrac{q^2}{c} - \dfrac{c}{q^2} = 2\Big(q - \dfrac{1}{q}\Big)$	asymptotic $h(n_1, n_2) \to q$
$c(1 + \alpha_i\beta_i) = q^3(\alpha_i + \beta_i)$	dispersion relation
$A_{ij} = A(\alpha_i, \alpha_j; c, q) = \dots$	phase shift

Semi-discretization ($\mathrm{dSK}_{l,m}$). The substitution

$$u = \frac{\tau_{m+l}\tau}{\tau_m \tau_l}, \quad f = \frac{\tau_m \tau_{-m}}{\tau^2}$$

brings ($\mathrm{dSK}_{l,m}$) to the bilinear equation

$$\tau_{l,t}\tau - \tau_l \tau_{,t} = \tau_m \tau_{l-m} - \tau_{-m}\tau_{l+m}$$

introduced in [16]. Again, the 2-soliton Ansatz

$$\tau = 1 + e_1 + e_2 + A_{12}e_1 e_2, \quad e_i = q_i^n \exp(-\omega_i t + \delta_i)$$

allows to find the dispersion law and the phase shift:

$$\omega_i = q_i^m - q_i^{-m}, \quad A_{ij} = \frac{(q_i^l - q_j^l)(q_i^m - q_j^m)}{(1 - q_i^l q_j^l)(1 - q_i^m q_j^m)}.$$

Then N-soliton solution satisfies the equation automatically.

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[16] X.B. Hu, P.A. Clarkson, R. Bullough. *J. Phys. A* ***30:20*** (1997) L669–676.

To compare with the continuous (SK) case [2, 17, 18]:

$$e_i = \exp(\kappa_i x - \omega_i t + \delta_i), \quad \omega_i = \kappa_i^5, \quad A_{ij} = \frac{(\kappa_i - \kappa_j)^2(\kappa_i^2 - \kappa_i\kappa_j + \kappa_j^2)}{(\kappa_i + \kappa_j)^2(\kappa_i^2 + \kappa_i\kappa_j + \kappa_j^2)}.$$

[17] P.J. Caudrey, R.K. Dodd, J.D. Gibbon. *Proc. Roy. Soc. London A* ***351*** (1976) 407–422.

[18] E. Date, M. Jimbo, M. Kashiwara, T. Miwa. *J. Phys. Soc. Japan* ***50*** (1981) 3813.

Breather-type solutions of (dSK$_{l,m}$) are obtained at

$$q_1 = \rho e^{\mathrm{i}\varphi}, \quad q_2 = \rho e^{-\mathrm{i}\varphi}, \quad \delta_1 = \alpha + \mathrm{i}\beta, \quad \delta_2 = \alpha - \mathrm{i}\beta.$$

Then

$$\omega = \mu + \mathrm{i}\nu, \quad \mu = (\rho^m - \rho^{-m})\cos m\varphi, \quad \nu = (\rho^m + \rho^{-m})\sin m\varphi,$$

$$A_{12} = -\frac{4\rho^{m+l}\sin l\varphi \sin m\varphi}{(1-\rho^{2l})(1-\rho^{2m})}$$

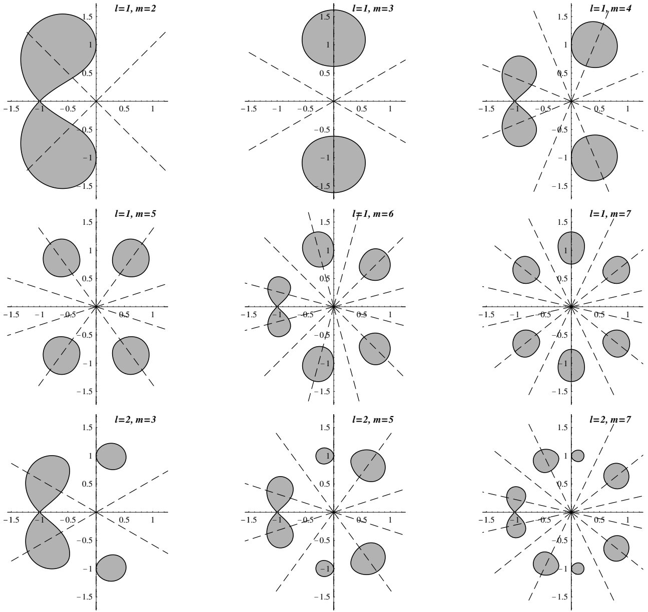
and 2-soliton solution takes the form

$$\tau = 1 + 2z\cos(\varphi n - \nu t + \beta) + A_{12}z^2, \quad z = \rho^n e^{\alpha - \mu t}.$$

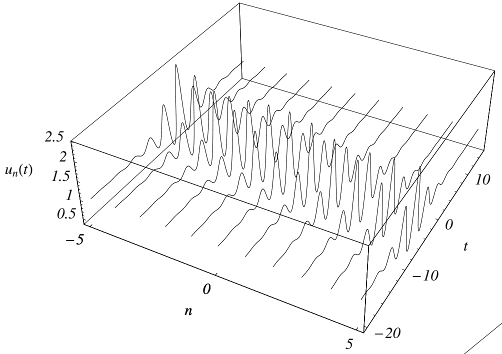
- Periodicity in t: $\mu = 0$, that is $\varphi = \frac{2k+1}{2m}\pi$.
- Regularity (of u): a restriction on the choice of q

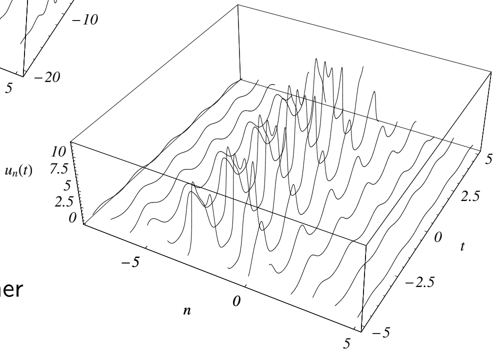
$$(\rho^l - \rho^{-l})(\rho^m - \rho^{-m}) + 4\sin l\varphi \sin m\varphi < 0$$

which gives several zones in $\mathbb{C}$.


l=1, m=2
l=1, m=3
l=1, m=4
l=1, m=5
l=1, m=6
l=1, m=7
l=2, m=3
l=2, m=5
l=2, m=7

a moving breather





a stationary breather